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AD-A210 315

EXTENDING MATCHINGS IN PLANAR GRAPHS IV

by

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dedicated to Gert Sabidussi on the occasion of his sixtieth birthday

ABSTRACT

The structure of certain non-2-extendable planar graphs is studied first. In particular, 4-connected 5-regular planar graphs which are not 2-extendable are investigated and examples of these are presented. It is then proved that all 5-connected even planar graphs are 2-extendable. Finally, a certain configuration called a *generalized butterfly* is defined and it is shown that 4-connected maximal planar even graphs which contain no generalized butterfly are 2-extendable.

1. Introduction and Terminology

Let p and n be integers with $0 < n < p/2$. A graph G is said to be **n-extendable** if G contains a matching of size n and every matching of size n extends to a perfect matching. In [10] (this paper will be considered part I of this series) it was proved that no planar graph is 3-extendable. On the other hand, many planar graphs are 1-extendable; for example, any 3-regular 2-line-connected graph is 1-extendable by a result of Berge [1, Theorem 13, pg. 160] and Cruse [2] (see also [8]). (Note that planarity is not a necessary part of the hypothesis here.) Thus perhaps the most interesting task remaining along this line is the study of 2-extendable planar graphs.

In paper II of this series [4], 2-extendability in the important class of *simple 3-polytopes* (i.e., 3-regular 3-connected planar graphs) was investigated. In particular, it was shown that any simple 3-polytope G having *cyclic connectivity* $c\lambda(G)$ at least 4 and having no faces of size 4 must be 2-extendable. If G is a *bipartite* simple 3-polytope with $c\lambda(G) \geq 4$, then by planar duality and Euler's theorem, G must contain faces of size 4. However, these graphs are 2-extendable. (This is an immediate corollary of Theorem 3.2 of [5].) It should be noted that planarity is quite crucial here. In [7] it is a corollary to a much more general result that there are graphs which are (non-planar) 3-regular 3-connected and not 2-extendable, but have arbitrarily large cyclic connectivity!)

In paper III of this series [6], 2-extendability of r-regular r-connected planar even graphs for $r = 4$ and 5 was investigated. (A graph is said to be even if it has an even number of points.) It was shown there that all 5-regular 5-connected planar graphs are

* work supported by ONR Contract #N00014-85-K-0488

2-extendable. The situation for the case when $r = 4$ is not so simple, but several sufficient conditions for 2-extendability were stated and proved in [6] as well.

In the case of graphs with connectivity less than 5, the presence of certain induced subgraphs clearly prevents 2-extendability. An important family of such subgraphs is defined as follows. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be two independent lines in a connected graph G. Then if the graph $G-u_1-v_1-u_2-v_2$ contains an odd component C_1, the induced subgraph $G[V(C_1) \cup \{u_1, v_1, u_2, v_2\}]$ is called a **generalized butterfly** (or **gbutterfly** in short). Obviously, if such a subgraph is present, the two lines e_1 and e_2 cannot be extended to a perfect matching and G is therefore not 2-extendable.

The absence of gbutterflies, however, is not enough to guarantee 2-extendability even in a 4-connected planar even graph. In Section 2 of the present paper, we investigate at some length the structure of 4-connected 5-regular planar even graphs without gbutterflies, but which still fail to be 2-extendable. A number of examples are presented as well.

In Section 3, it is shown that 5-connected planar even graphs are 2-extendable whether or not they are regular. Finally, it is proved that a 4-connected maximal planar even graph containing no gbutterfly must be 2-extendable.

Note that all graphs in this paper are assumed to be connected, unless otherwise specified. For the sake of brevity, we shall abbreviate n-regular by **nR**, n-connected by nC, even by **E**, planar by **P** and maximal planar by **MAXP**. For example, "G is 4C5REP" means "G is 4-connected, 5-regular, even and planar". We abbreviate "perfect matching" by **pm**. Also, if S is a cutset of points in a connected graph G, denote by $c_o(G - S)$ the number of odd components of $G - S$. If F is a face of a plane graph we shall denote the cycle bounding this face by ∂F. Finally, if points u and v of a graph are adjacent, we shall often write $u \sim v$.

2. Properties of an Exceptional Family of Graphs

A property of graphs different from 2-extendability, but nevertheless, closely related to it, is that of *bicriticality*. A graph G is **bicritical** if $G - u - v$ has a pm for all pairs of distinct points u and v. A 3-connected bicritical graph is called a **brick**. Bricks play an important role in a canonical decomposition of graphs in terms of their matchings. (See [8] for details.) The family of 2-extendable graphs partitions nicely in that such a graph is either bipartite or bicritical (Theorem 4.2 of [9]). (That no graph can be both bipartite and bicritical is immediate.)

Theorem 2.1. If G is 4CPE, then G is a brick.

Proof. Choose $u, v \in V(G)$. By a result of Thomassen (Corollary 2 of [11]), there is a Hamilton path π joining u and v. (This also follows from Tutte's theorem on Hamiltonian cycles in planar graphs [13] when used in its full generality. This theorem of Tutte is in turn a corollary of Thomassen's main result in [11].) But path π has odd length and hence so does $\pi - u - v$. Thus $\pi - u - v$ contains a pm of $G - u - v$. So G is bicritical and since it is 4-connected, it is a brick. ∎

But, since all bicritical graphs are 1-extendable, we then have the following immediate corollary.

Corollary 2.2. If G is 4CPE, then G is 1-extendable. ∎

It is not true in general, however, that if G is 4CEP, then G is 2-extendable. If fact, there are examples of 4CEP graphs which are 4-regular or 5-regular, have no gbutterflies and yet are still not 2-extendable. For the 4-regular case, the reader is referred to [6]. We will study the 5-regular case below.

At this point let us introduce the concept of an $\{e_1, e_2\}$-*blocker*. Let G be any connected even graph and let e_1 and e_2 be any two independent lines in G. A set $S \subseteq V(G)$ is an $\{e_1, e_2\}$-**blocker** if S contains both lines e_1 and e_2 and $|S| = c_o(G - S) + 2$.

Lemma 2.3. Suppose graph G is 1-extendable, but not 2-extendable. Suppose, in particular, that $\{e_1, e_2\}$ does not extend. Then G contains an $\{e_1, e_2\}$-blocker.

Proof. Let $e_i = u_i v_i$, $i = 1, 2$. Now $G'' = G - u_1 - v_1 - u_2 - v_2$ has no pm, so by Tutte's 1-factor theorem [12], there is a set $S'' \subseteq V(G'')$ such that $|S''| < c_o(G'' - S'')$ and hence by parity (since G is even), $|S''| \leq c_o(G'' - S'') - 2$. But G is 1-extendable and so e_1 lies in a pm of G. Thus $|S''| = c_o(G'' - S'') - 2$ and so if we let $S = S'' \cup \{u_1, v_1, u_2, v_2\}$, we have $|S| = |S''| + 4 = c_o(G'' - S'') + 2 = c_o(G - S) + 2$ and S is an $\{e_1, e_2\}$-blocker. ∎

Although we make no further use of it in the present paper, we include the next result on *minimal* blockers. A set $S \subseteq V(G)$ is a **minimal** $\{e_1, e_2\}$-blocker if it is an $\{e_1, e_2\}$-blocker, but no proper subset of S is a blocker with respect to this same pair of lines $\{e_1, e_2\}$.

Theorem 2.4. Let G be a 1-extendable graph containing the two independent lines $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$ and let S be a minimal $\{e_1, e_2\}$-blocker. Let $S'' = S - u_1 - v_1 - u_2 - v_2$. Then, if $S'' \neq \emptyset$, each point of S'' is adjacent to no odd component of $G - S$ or to at least three odd components of $G - S$.

Proof. Suppose $S'' \neq \emptyset$. Suppose $u \in S''$ and suppose u is adjacent to at least one odd component of $G - S$.

Suppose now that $u \in S''$ is adjacent to exactly one odd component of $G'' - S''$, say C_1. Then $G[V(C_1) \cup \{u\}]$ is an even component of $G'' - S''$. So $c_o(G'' - (S'' - u)) = c_o(G'' - S'') - 1$. So $|S'' - u| \leq |S''| - 1 = c_o(G'' - S'') - 2 - 1 = c_o(G'' - (S'' - u)) - 1 - 1 = c_o(G'' - (S'' - u)) - 2$ and again since G has a pm, equality must hold. Hence the minimality of S is contradicted.

Now suppose that $u \in S''$ is adjacent to exactly two odd components C_1 and C_2 of $G - S$. Let $S''' = S'' - u$. Then $G[V(C_1) \cup V(C_2) \cup \{u\}]$ is an odd component of $G'' - S'''$, so $c_o(G'' - S''') = c_o(G'' - S'') - 1$. Thus $|S'''| = |S''| - 1 = c_o(G'' - S'') + 2 - 1 = c_o(G'' - S''') + 1 + 2 - 1 = c_o(G - S''') + 2$, again contradicting the minimality of S. ∎

We note in passing that regardless of whether $\{e_1, e_2\}$-blocker S is minimal or not, each of the four points u_1, v_1, u_2 and v_2 must be adjacent to at least one of the odd components of $G - S$ since G is 1-extendable. Also note that for any blocker S, no pair of independent lines contained in $G[S]$ can extend to a pm; not just the pair $\{e_1, e_2\}$ used to define the blocker.

For the remainder of this paper, let us call any graph G **exceptional** if it is 4CPE, but not 2-extendable.

Theorem 2.5. Suppose G is exceptional, e_1 and e_2 are two independent lines in $E(G)$ and S is an $\{e_1, e_2\}$-blocker. Then:

(a) $G - S$ has no even components and

(b) each odd component of $G - S$ has exactly four points of attachment in S.

Proof. Form a new graph BG from G by contracting all components (odd or even) to singletons and then choose, for each odd component C_i of $G - S$, four lines from C_i to four different points of S. (Note that this is possible since G is 4-connected.) Finally, delete all lines in $G[S]$ and all singletons which correspond to *even* components of $G - S$. Let the new bipartite graph thus formed be denoted BG. Graph BG is planar since G is and has as its bipartition $S \cup \{\widehat{C_1}, \ldots, \widehat{C_{s-2}}\}$.

Now $|E(BG)| = 4(s-2) = 4s-8$. But by Euler's theorem, $|E(BG)| \leq 2|V(BG)| - 4 = 4s-8$. Thus equality must hold and it follows that BG is a *maximal* bipartite planar graph so all faces of BG must be quadrilaterals. Now reinsert one of the *even* components of $G - S$, call it R, while maintaining planarity. It follows that R must fit into the interior of one of the quadrilateral faces of BG. But R has no lines to any odd component C_i of $G - S$ and hence can have lines only to the two points of the quadrilateral which belong to S. But this contradicts the fact that G is 4-connected. Hence there can be no even components of $G - S$.

To prove part (b), note that since graph BG is maximal bipartite planar, no additional line from any point of an odd C_i to a fifth point of S can be reinserted without destroying planarity.

∎

The object of the next several results is to discuss the structure of exceptional graphs which are, in addition, 5-regular and then to produce examples of such graphs.

Theorem 2.6. If G is a 5-regular exceptional graph and S is any $\{e_1, e_2\}$-blocker in G, then:

(a) the induced subgraph $G[S]$ contains 2,3,4 or 5 lines,

(b) no component of $G - S$ is a singleton, and

(c) if C_i is any odd component of $G - S$, then C_i is attached to S by 5,7,9 or 11 lines.

Proof. Recall from Theorem 2.5(a) that $G - S$ has no even components. Let $s = |S|$ and let N be the number of lines with precisely one endpoint in S.

Now viewed from S, $N \leq 5(s-4) + 16 = 5s - 4$, while viewed from the odd components of $G - S$, $N \geq 5(s-2) = 5s - 10$ and part (a) follows.

Part (b) follows immediately from 5-regularity and Theorem 2.5(b).

To prove part (c), note that $2q_i = \sum_{v \in C_i} \deg_{C_i} v = \sum_{v \in C_i} \deg_G v - N_i = 5|V(C_i)| - N_i$, where N_i is the number of lines joining C_i to S and $q_i = |E(C_i)|$. But $|V(C_i)|$ is odd and hence by parity, N_i is odd. Since G is 4-connected, $N_i \geq 5$. If $N_i \geq 13$, then $N \geq 13 + 5(s-3) = 5s - 2 > 5s - 4$, contradicting the inequality obtained in the proof of part (a). ∎

Next we obtain a lower bound on the size of the odd components in $G - S$ when S is an $\{e_1, e_2\}$-blocker.

Theorem 2.7. If G is a 5-regular exceptional graph, e_1 and e_2 are two independent lines in G, S is an $\{e_1, e_2\}$-blocker and if the odd components of $G - S$ are $C_1, \ldots, C_{s-2}$, then $|V(C_i)| \geq 11$ for all i, $1 \leq i \leq |S| - 2$.

Proof. From Theorem 2.6, we know that $|V(C_i)| \geq 3$, for all i.

First suppose $|V(C_i)| = 3$. Then by 5-regularity, component C_i sends at least nine lines to S. Since C_i is connected, it contains at least one point of degree 2 in C_i. Let u_1 be such a point. Furthermore, let $V(C_i) = \{u_1, u_2, u_3\}$ and let $\{x_1, x_2, x_3, x_4\}$ be the four points of attachment for C_i in S. Now u_1 is adjacent to exactly three points of S, say, without loss of generality, to x_1, x_2 and x_3. Also u_2 must be adjacent to at least three of the four x_i's and hence to at least two of x_1, x_2 and x_3. Now permuting the labels of x_1, x_2, x_3 if necessary, without loss of generality we may assume that u_2, x_1, x_2, x_3 is the clockwise order of these four points about point u_1. Then if $u_2 \sim x_1$ and x_2, G contains a separating triangle. Similarly, if $u_2 \sim x_2$ and x_3 or $u_2 \sim x_1$ and x_3. But this contradicts the 4-connectedness of G.

So no C_i contains precisely three points and hence $|V(C_i)| \geq 5$. Let $c_i = |V(C_i)|$ and (as before) let N_i denote the number of lines from C_i to S.

1. Suppose $N_i = 5$.

Then $\sum_{v \in C_i} \deg_{C_i} v = 2q_i = 5c_i - 5$. On the other hand, by planarity, $q_i \leq 3c_i - 6$ and so $5c_i - 5 = 2q_i \leq 6c_i - 12$ and thus $c_i \geq 7$.

1.1. Suppose $c_i = 7$.

Then equality holds in the preceding inequality and it follows that C_i is MAXP.

We know that at least two points of C_i send no line to S and hence are of degree 5 in C_i. Let u be one such point and let $y_1, \ldots, y_5$ be the five neighbors of u in clockwise order. Since C_i is MAXP, the five faces at u are triangles and hence $y_1 \cdots y_5 y_1$ is a 5-cycle Z. If Z does not separate u from S, then triangle uy_1y_2u separates y_3, y_4 and y_5 from S, a contradiction of 4-connectedness. So Z separates u from S. Since G contains no separating triangles, the seventh point of C_i – call it z – is separated from u by cycle Z. If at least three lines join z to S, then component C_i must contain a point cutset of G of size no greater than 3, a contradiction. So no more than two lines join z to S. But then at least three lines must join z to Z and hence some two of these lines must form two sides of a quadrilateral Q through u. But then Q separates one point on Z from two others and it then follows that Z must contain a point of degree in G no greater than 4, a contradiction.

1.2. Suppose $c_i = 9$.

Then $\sum_{v \in C_i} \deg_{C_i} v = 5 \cdot 9 - 5 = 40 = 2q_i$ and hence $q_i = 20$. On the other hand, $3c_i - 6 = 21$, so C_i must have exactly one quadrilateral face F_4 and all its remaining faces must be triangles. Since G is 4-connected, all triangular faces of C_i must be triangular faces in G as well. So we may suppose that ∂F_4 separates all of $G - V(C_i)$ from all five points of C_i not on ∂F_4. Now ∂F_4 separates the plane into two regions. Without loss of generality, let us call the region containing the other five points of G the "interior" of ∂F_4 and the other (which contains all of $G - V(C_i)$) the "exterior" of ∂F_4. So ∂F_4 contains three points each sending one line to S and the fourth sending two lines to S, while interior

to ∂F_4 lie the other five points of C_i each of degree 5 in C_i. Let ∂F_4 be $w_1w_2w_3w_4w_1$ (clockwise) where without loss of generality, we may assume $\deg_{C_i} w_1 = 3$. Let x be the only neighbor of w_1 interior to ∂F_4. Then all five faces at x are triangles and hence $x \sim w_2$ and $x \sim w_4$. Moreover, $x \not\sim w_3$ since G contains no separating triangle.

So let y, z be the fourth and fifth neighbors of x interior to ∂F_4, where the neighbors of x are (clockwise) w_1, w_2, y, z and w_4. Then $w_2 \sim y \sim z \sim w_4$. But then the remaining two points inside pentagon $w_4zyw_2w_3w_4$ cannot be adjacent to either w_2 or w_4 by 5-regularity. Thus $\{z, y, w_3\}$ contains a cutset of G, contradicting 4-connectivity.

2. Suppose $N_i = 7$.

2.1. Suppose also that $c_i = 5$.

So $\sum_{v \in C_i} \deg_{C_i} v = 2q_i = 5 \cdot 5 - 7 = 18 \le 2(3c_i - 6) = 2(3 \cdot 5 - 6) = 18$. Thus equality holds and C_i is MAXP. So when G is drawn in the plane some non-empty part of $G - V(C_i)$ is separated from some other points of C_i by a separating triangle, thus contradicting 4-connectivity.

2.2. Now suppose that $c_i = 7$.

Thus $\sum_{v \in C_i} \deg_{C_i} v = 2q_i = 7 \cdot 5 - 7 = 28$, so $q_i = 14$, while $3c_i - 6 = 15$. Arguing in a manner similar to Case 1.2, we may assume that C_i contains a quadrilateral face F_4 such that all faces of C_i interior to ∂F_4 are triangles, while all of $G - V(C_i)$ lies exterior to ∂F_4. Again label ∂F_4 clockwise by $w_1w_2w_3w_4w_1$. By 4-connectivity, each w_i has at least one neighbor interior to ∂F_4 and at least one exterior to ∂F_4. Then by 5-regularity, we may suppose, without loss of generality, that each of w_1, w_2 and w_3 sends two lines to the exterior and w_4 sends one. Hence w_1, w_2 and w_3 each send one line to the interior of ∂F_4 and w_4 sends two.

Let w_5, w_6 and w_7 be the remaining three points of C_i. Without loss of generality, suppose $w_5 \sim w_2$. Then since the 2 interior faces at w_2 must be triangles, $w_5 \sim w_1$ and $w_5 \sim w_3$ and it follows that w_6 and w_7 are interior to quadrilateral $w_1w_5w_3w_4w_1$. But then $\{w_4, w_5\}$ must contain a cutset of G, yet again contradicting 4-connectivity.

2.3. Suppose $c_i = 9$.

Arguing as before, C_i is such that the addition of two more lines results in a MAXP graph. So any imbedding of G (and hence of C_i) in the plane must result in (a) all triangular faces for C_i , except two, which must be quadrilaterals or (b) all triangular faces for C_i except one which must be a pentagon.

First, suppose that all faces of C_i are triangles, except for two quadrilateral faces F_1 and F_2. Since G is 4-connected, at least one of F_1 and F_2 – say F_2 – has the property that ∂F_2 sends at least four lines to some component H_i of $G - V(C_i)$. But again by 4-connectivity, all seven lines counted by N_i must join ∂F_2 to H_i and hence in fact H_i is all of $G - V(C_i)$. Let us agree to call the region of the plane determined by ∂F_2 and which contains $G - V(C_i)$ the "exterior" of ∂F_2. Then all of the rest of C_i lies interior to ∂F_2 and ∂F_1 bounds a quadrilateral face in this region.

Let ∂F_2 be represented (in clockwise order) as $w_1w_2w_3w_4w_1$. Since C_i sends seven lines to S and by 4-connectivity we may assume without loss of generality that w_1, w_2 and w_3 send two lines each to S and w_4 sends one line to S. Hence by 5-regularity, w_1, w_2 and w_3 each send one line to S and w_4 sends two such lines.

Suppose w_1 also lies on ∂F_1. Suppose also that w_2 lies on ∂F_1. In fact, let us say that

$\partial F_1 = w_1w_2y_1y_2w_1$ (clockwise). Then $w_4 \sim y_2$ and triangle $y_2w_4w_1y_2$ is the boundary of a face in G. Similarly, $w_3 \sim y_1$ and triangle $y_1w_2w_3y_1$ is a face in G as well. But then the three remaining points of C_i lie interior to the quadrilateral $y_1w_3w_4y_2y_1$ and since $\deg_G w_3 = 5$, the set $\{y_1, y_2, w_4\}$ contains a cut of G. Again 4-connectivity is violated.

So we may suppose w_2 does not lie on ∂F_1 and hence w_4 lies on ∂F_1. Say $\partial F_1 = w_1y_1y_2w_4w_1$ (clockwise). Then $w_2 \sim y_1$ and triangle $w_2w_3y_1w_2$ is a face boundary. Thus $w_3 \sim y_1$ and triangle $w_2w_3y_1w_2$ is also a face boundary. But then the remaining three points of C_i lie interior to the quadrilateral $y_1w_3w_4y_2y_1$ and hence $\{y_1, y_2, w_4\}$ contains a cutset of G, again a contradiction.

Hence w_1 does not lie on ∂F_1 (and by symmetry, w_3 does not lie on ∂F_1 either). Let z be the neighbor of w_1 which lies interior to ∂F_2. Then $z \sim w_2$ and triangle $w_1w_2zw_1$ is a face boundary. But then the four remaining points of C_i lie interior to pentagon $w_1zw_2w_3w_4w_1$ and hence $\{w_3, w_4, z\}$ contains a cut of G, a contradiction.

So we may suppose that all faces of C_i are triangular, except one pentagonal face F_5. Since G is 4-connected, at least four points on ∂F_5 send lines to S. (Let us call the region of the plane containing these lines to S the "exterior" of ∂F_5.) Let the points of ∂F_5 be $w_1, \ldots, w_5w_1$ in clockwise order.

2.3.1. Suppose there are exactly four points on ∂F_5 sending lines to $G - V(C_i)$. Without loss of generality, assume $\deg_{C_i} w_1 = 5$. Let the neighbors of w_1 be w_2, x_1, x_2, x_3, w_5 (clockwise). Then triangles $w_1w_2x_1w_1$, $w_1x_1x_2w_1$, $w_1x_2x_3w_1$ and $w_1x_3w_5w_1$ are all face boundaries in G. Since G is 4-connected, $x_1 \not\sim x_3, w_5$.

2.3.1.1. Suppose $x_1 \sim w_3$.

Then by 4-connectivity, triangle $w_2w_3x_1w_2$ is a face boundary. Let x_4 be the fifth neighbor of x_1.

Suppose x_4 is interior to the hexagon $w_3w_4w_5x_3x_2x_1w_3$ and hence is the ninth point of C_i. So $w_3 \sim x_4 \sim x_2$ and triangles $x_1w_3x_4x_1$ and $x_1x_4x_2x_1$ are face boundaries. But then $x_4 \sim w_4$ and triangle $x_4w_3w_4x_4$ is also a face boundary.

But now consider the fifth neighbor of x_2. If $x_2 \sim w_4$, then $\deg_G x_4 = 4$, while if $x_2 \sim w_5$, then $\deg_G x_3 = 3$. In either case we have a contradiction.

So we may assume that $x_4 = w_4$. Then $\deg_G x_1 = 5$ implies $x_2 \sim w_4$ and triangles $x_1w_4x_2x_1$ and $x_1w_3w_4x_1$ are face boundaries. But then the ninth point of C_i must lie in the interior of quadrilateral $w_4w_5x_3x_2w_4$ and hence $\{w_5, x_2, x_3\}$ contains a cut of G, a contradiction.

2.3.1.2. So suppose $x_1 \not\sim w_3$. Then $x_1 \sim w_4$ and $x_1 \sim x_4$ where x_4 is the fifth neighbor of x_1 and the ninth point in C_i. Now since $\deg_G x_4 = 5$, x_4 is not interior to the quadrilateral $w_2w_3w_4x_1w_2$. Thus x_4 is interior to pentagon $w_4w_5x_3x_2x_1w_4$ and must be adjacent to all five of these points. But then $\deg_G x_3 = 4$, a contradiction.

2.3.2. So suppose all five points on ∂F_5 send lines to S. Since $N_i = 7$, we may suppose without loss of generality that w_1 sends exactly one line to S.

Suppose that $w_1 \sim w_3$. Then if $w_1 \sim w_4$, triangles $w_1w_2w_3w_1$, $w_1w_3w_4w_1$ and $w_1w_4w_5w_1$ are all face boundaries. But then $N_i = 11$, a contradiction. So $w_1 \not\sim w_4$. Then the fifth neighbor of w_1 – call it x_1 – must lie interior to quadrilateral $w_1w_3w_4w_5w_1$ and $w_3 \sim x_1 \sim w_5$ and triangles $w_1w_2w_3w_1$, $w_1w_3x_1w_1$ and $w_1x_1w_5w_1$ are all face boundaries. But then $\{x_1, w_4, w_5\}$ must contain a cutset of G, a contradiction.

So we may suppose that $w_1 \not\sim w_3$ and by symmetry, that $w_1 \not\sim w_4$. Thus the fourth and fifth neighbors of w_1 – call them x_1 and x_2 – must lie interior to ∂F_5 and we may suppose that $w_2 \sim x_1 \sim x_2 \sim w_5$ and all triangles at w_1 are face boundaries.

Suppose w_2 sends two lines to S. Thus $x_1 \sim w_3$ and triangle $w_2w_3x_1w_2$ is a face boundary.

Suppose $x_1 \sim w_4$. Then the eighth and ninth points of C_i must lie interior to the quadrilateral $x_1w_4w_5x_2x_1$ and hence $\{w_4, w_5, x_2\}$ contains a cutset of G, a contradiction.

So $x_1 \not\sim w_4$. By 4-connectivity, $x_1 \not\sim w_5$, so if x_3 is the fifth neighbor of x_1, then x_3 lies in the interior of the pentagon $x_1w_3w_4w_5x_2x_1$ and $\deg_G x_3 = 5$ implies that $w_3 \sim x_3 \sim x_2$ and the five triangles at x_1 are face boundaries. Also $\deg_G w_3 = 5$ implies $x_3 \sim w_4$ and triangle $w_3w_4x_3w_3$ is a face boundary. Thus if x_4 is the ninth point of C_i, it must lie interior to quadrilateral $x_2x_3w_4w_5x_2$ and hence $\deg_G x_4 \leq 4$, a contradiction.

So we may suppose that w_2 sends exactly one line to S and by symmetry, that w_5 sends exactly one line to S.

Suppose $w_2 \sim w_4$. Then $w_4 \sim x_1$ and the remaining two points of C_i must lie interior to quadrilateral $x_1w_4w_5x_2x_1$. But $\deg_G w_4 = 5$ implies that $\{w_5, x_2, x_1\}$ contains a cutset of G, a contradiction.

So we may assume that $w_2 \not\sim w_4$. By 4-connectivity, $w_2 \not\sim x_2, w_5$, so the fifth neighbor x_4 of w_2 lies interior to hexagon $w_2w_3w_4w_5x_2x_1w_2$. Then $w_3 \sim x_4 \sim x_1$ and all triangles at w_2 are face boundaries. Consider the fifth neighbor of x_1 – call it x_5. Then $x_5 \not\sim w_5, w_3$ since G contains no separating triangles.

Suppose $x_5 = w_4$. Then $x_4 \sim w_4 \sim x_2$ and the ninth point of C_i lies interior to one of the four triangles $w_3w_4x_4w_3$, $x_4w_4w_1x_4$, $x_1w_4x_2x_1$ or $x_2w_4w_5x_2$. But then G is not 4-connected, a contradiction.

Hence $x_5 \neq w_4$. Thus x_5 must lie interior to hexagon $x_1x_4w_3w_4w_5x_2x_1$ and $x_4 \sim x_5 \sim x_2$. Suppose $x_2 \sim w_4$. Then $w_4 \sim x_5$ and $x_5 \sim w_3$. But then $\deg_G x_4 = 4$, a contradiction. So $x_2 \not\sim w_4$. Since there are no separating triangles, $x_2 \sim w_3$. But then $\deg_G w_3 = 5$ implies that $x_2 \sim x_4$ and hence triangle $x_1x_4x_2x_1$ is a separating triangle, a contradiction.

3. Finally, suppose $N_i = 9$.

At this point, we may suppose all C_i's have at least five points and at least nine lines each to S. Thus $9(s-2) \leq N \leq 5(s-4) + 4 \cdot 4$ or $9s - 18 \leq N \leq 5s - 4$. So $4s \leq 14$ and $s < 4$, contradicting the 4-connectivity of G and completing the proof of the theorem. ∎

Corollary 2.8. If G is a 5-regular exceptional graph then $|V(G)| \geq 26$ and if, in addition, G does not contain a gbutterfly, $|V(G)| \geq 38$.

Proof. Let S be an $\{e_1, e_2\}$-blocker and let $s = |S|$. Then by Theorem 2.7, $11(s - 2) + s \leq |V(G)|$; that is, $12s - 22 \leq |V(G)|$. But $s \geq 4$, so the first result follows. If G does not contain a gbutterfly, then $s \geq 5$ and the second result is proved. ∎

We make no claim that either of the bounds in the above corollary is sharp.

We now present four exceptional graphs which, in addition, are all 5-regular and gbutterfly-free. Our examples make use of the four graphical fragments labeled A, B, C and D and displayed in Figures 1 – 4. Note that these fragments have 5, 7, 9 and 11

lines of attachment respectively. These four values are the only possibilities for 5-regular exceptional graphs by the preceding theorem.

Figure 1. Fragment A

Recall from Theorem 2.6(a) that if G is a 5-regular exceptional graph and if S is any $\{e_1, e_2\}$-blocker in G, then $G[S]$ contains 2, 3, 4 or 5 lines. In Figures 5, 6, 7 and 8, we produce examples in which $G[S]$ contains these four allowed numbers of lines respectively. In all four examples, the small dark points are the points of S and the rest of the points of each graph are found in fragments of types A, B, C and D defined above. The four graphs in Figures 5 – 8 have 194, 180, 184 and 170 points respectively. Finally, note that none of these four graphs contains any gbutterflies.

Figure 2. Fragment *B*

Figure 3. Fragment *C*

Figure 4. Fragment ***D***

Figure 5. An exceptional graph with 2 lines in $G[S]$

Figure 6. An exceptional graph with 3 lines in $G[S]$

Figure 7. An exceptional graph with 4 lines in $G[S]$

Figure 8. An exceptional graph with 5 lines in $G[S]$

3. Two Classes of Planar 2-extendable Graphs

We now present the first of two general classes of planar graphs which *are* 2-extendable. The first result generalizes Theorem 2 of [6].

Theorem 3.1. If G is 5CPE, then G is 2-extendable.

Proof. By Corollary 2.2, G is 1-extendable. Suppose that e_1 and e_2 are two independent lines which do not extend to a pm. Then G is exceptional and by Lemma 2.3, graph G contains an $\{e_1, e_2\}$-blocker S. But by Theorem 2.5(b), G has a cutset of size no greater than 4, a contradiction. ∎

We now turn to our second class of 2-extendable planar graphs. Recall that a plane graph G is **maximal planar (MAXP)**, if all faces of G are triangles.

Theorem 3.2. If G is 4CMAXPE with no gbutterflies, then G is 2-extendable.

Proof. Let G be as in the hypothesis and suppose independent lines e_1 and e_2 do not extend. Note that by Corollary 2.2, G is 1-extendable. So G is exceptional and by Lemma 2.3, contains an $\{e_1, e_2\}$-blocker S. Moreover, by Theorem 2.5(a), graph $G - S$ contains no even component.

Form a simple maximal planar bipartite graph BG just as in the proof of Theorem 2.5.

Consider $\widehat{C_1}$ and its four neighbors in BG. Call these neighbors w_1, w_2, w_3, w_4 (clockwise about $\widehat{C_1}$). Let $\widehat{C_i}$ be the fourth point of the face the boundary of which contains w_1, w_2 and $\widehat{C_1}$. Let $\widehat{C_j}$ be the fourth point of the face containing points w_3, w_4 and $\widehat{C_1}$.

First suppose $\widehat{C_i} = \widehat{C_j}$. (See Figure 9(a).) Now no lines of G can join C_1 and C_j. Moreover, since $V(BG)$ contains S and is maximal bipartite planar, no point of S can lie interior to a face of BG. Thus, since G is MAXP, it follows that $w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_1$. But then G contains a gbutterfly; for example, the subgraph consisting of C_1, points w_1, w_2, w_3 and w_4, the lines of BG joining these four points to C_1 and the two lines $w_1 w_2$ and $w_3 w_4$.

If $\widehat{C_i} \neq \widehat{C_j}$, (see Figure 9(b)), the argument is essentially the same. ∎

Figure 9

Several remarks are in order at this point.

Remark 1. First we note that Theorem 3.2 is not a corollary of Theorem 3.1. The graph in Figure 10 is 4CMAXPE and has no gbutterflies, so it is 2-extendable. But the graph is *not* 5-connected. Note that if the four endpoints of lines e_1 and e_2 are deleted, two *even* components remain.

Figure 10

Remark 2. There do exist graphs which are 4CMAXPE, which contain gbutterflies and hence are not 2-extendable. The first two members $J(22)$ and $J(36)$ of an infinite family $J = \{J(22+14k)\}_{k=0}^{\infty}$ where $|J(22+14k)| = 22+14k$, are shown in Figure 11. (It should be clear to the reader how, for $k \geq 2$, to construct $J(22+14k)$, given $J(22+14(k-1))$.) In each member of this infinite family, lines e_1 and e_2 do not extend to a pm.

Figure 11

Remark 3. There are graphs which are 3CMAXPE, but do not even contain a pm! To provide an infinite family of such graphs, we use the concept of a *Kleetope*. (See Grünbaum [3].)

For $r \geq 3$, let $T(2r)$ denote the maximal planar graph on $2r$ points shown in Figure 12. Now construct the Kleetope over $T(2r)$, $Kl(T(2r))$, which is the graph obtained from $T(2r)$ by inserting a new "red" point in the interior of each triangular face of $T(2r)$ and joining the red point to each of the three points bounding the face in which it lies. Since $T(2r)$ has $4r-4$ faces, the Kleetope $Kl(T(2r))$ is a triangulation having $2r+(4r-4)=6r-4$ points and hence is even. But $Kl(T(2r))$ has an independent set of size $4r-4$ (namely, the set of "red" points) and $4r-4 > |V(Kl(T(2r)))|/2$, so $Kl(T(2r))$ has no pm.

Figure 12

Remark 4. If we try to weaken the hypothesis of Theorem 3.2 in yet another way, again we lose 2-extendability. A graph G is said to be **triangular** if every line of G is a line of some triangle in G. In Figure 13 we display an example of a graph G which is 4CPE, is triangular and contains no gbutterflies, but is *not* 2-extendable.

Note that the large points labeled with a "B" denote fourteen instances of substituting the 17-point subgraph shown. The resulting graph G has $17 \times 14 + 16 = 254$ points and mindeg $G = 5$. However, no two of the four lines $e_1, \ldots, e_4$ extend to a pm. This is easy to see, for if one deletes the four endpoints of, say, e_1 and e_2, there remains a graph G'' with a set S'' of 12 points (i.e., the points not labeled "B") and such that $G'' - S''$ has 14 odd components (i.e., the 14 components labeled "B"). Since $|S''| < c_o(G'' - S'')$, G'' has no pm and hence G is not 2-extendable.

We observe that one can construct an infinite family of graphs (of which the graph in Figure 13 is the smallest) all of which are 4CPE, triangular, have no gbutterflies and are not 2-extendable by suitably enlarging the subgraph denoted by "B". The details are left to the reader.

Figure 13

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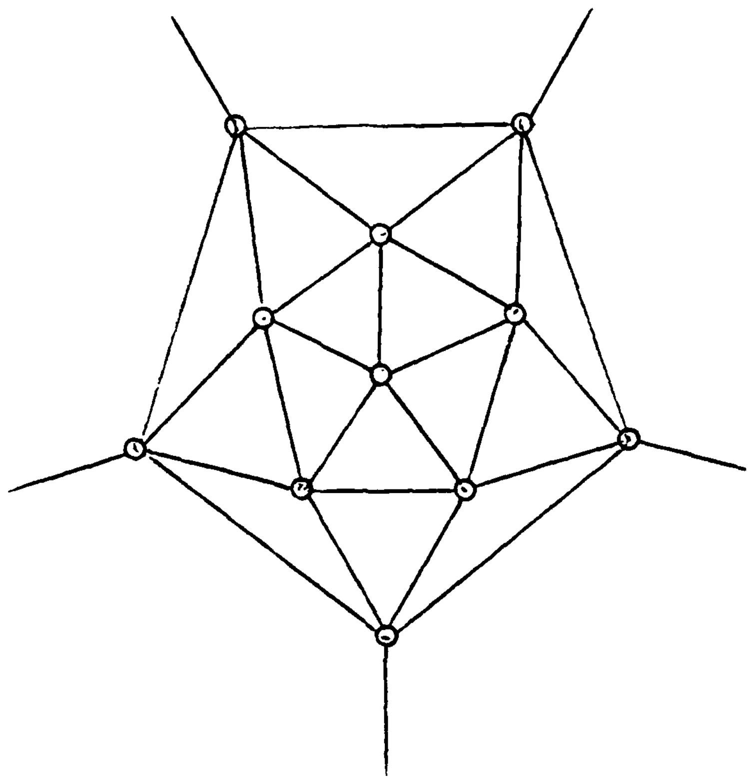


Figure 1. Fragment A

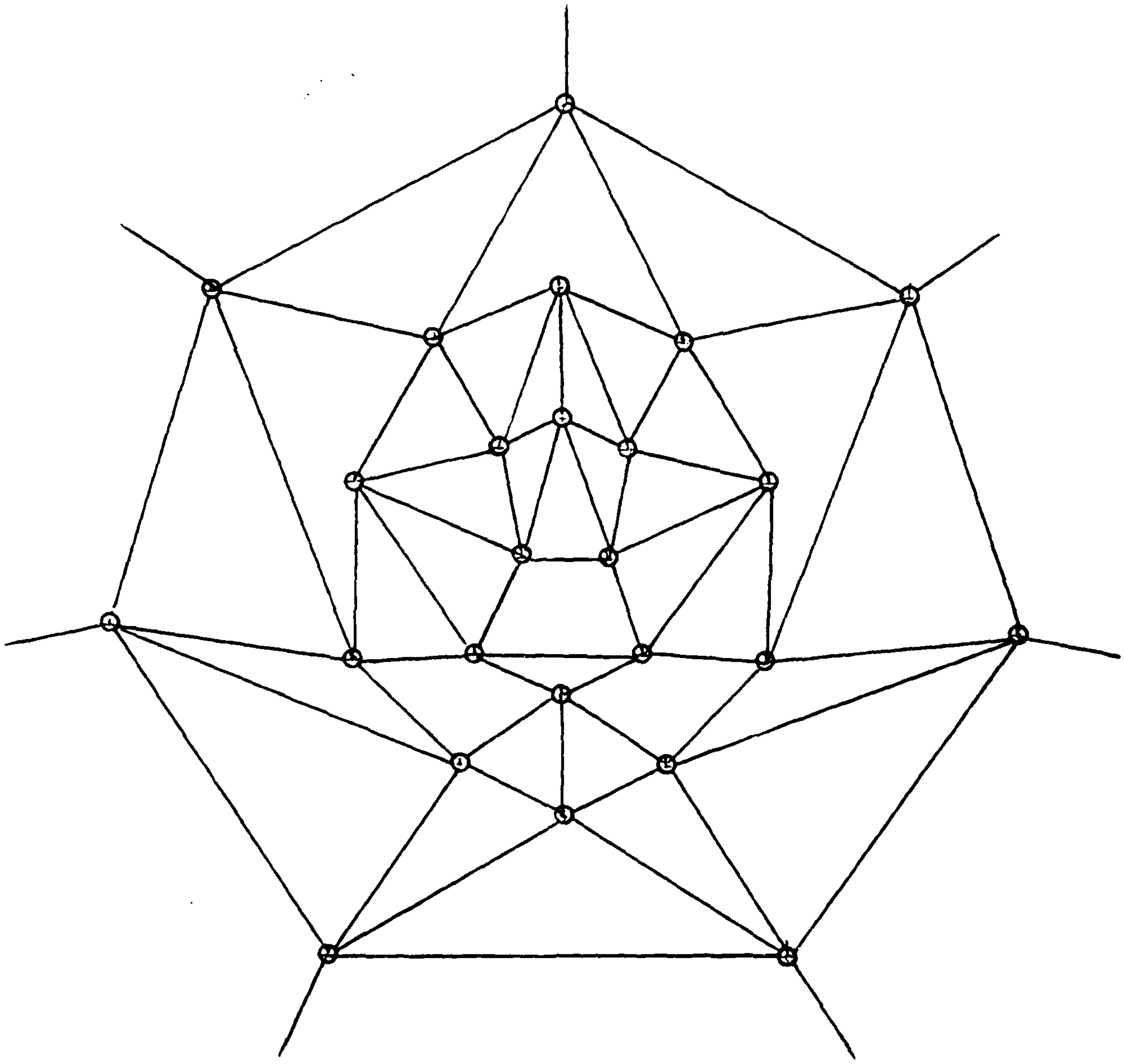


Figure 2. Fragment B

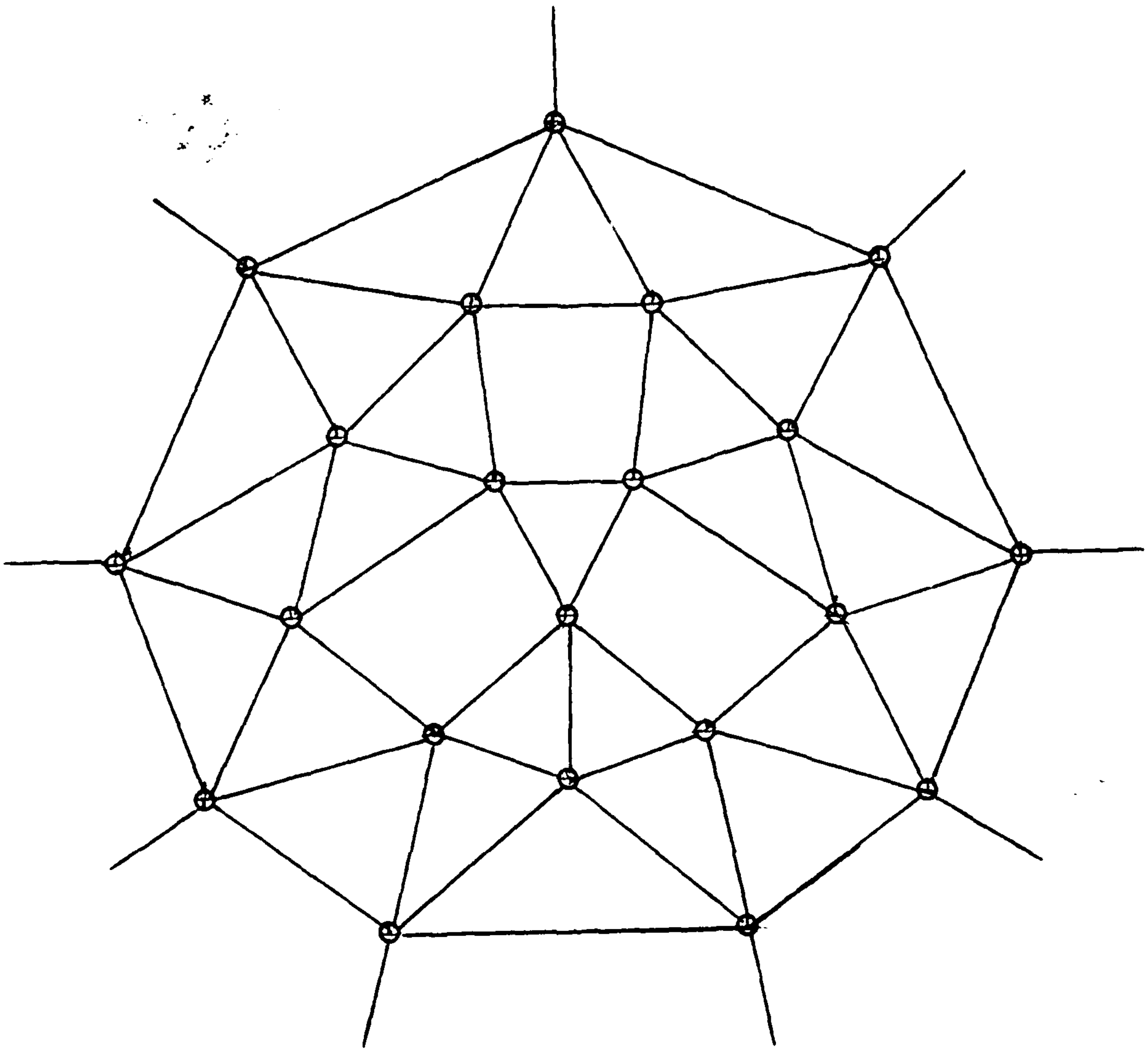


Figure 3. Fragment C

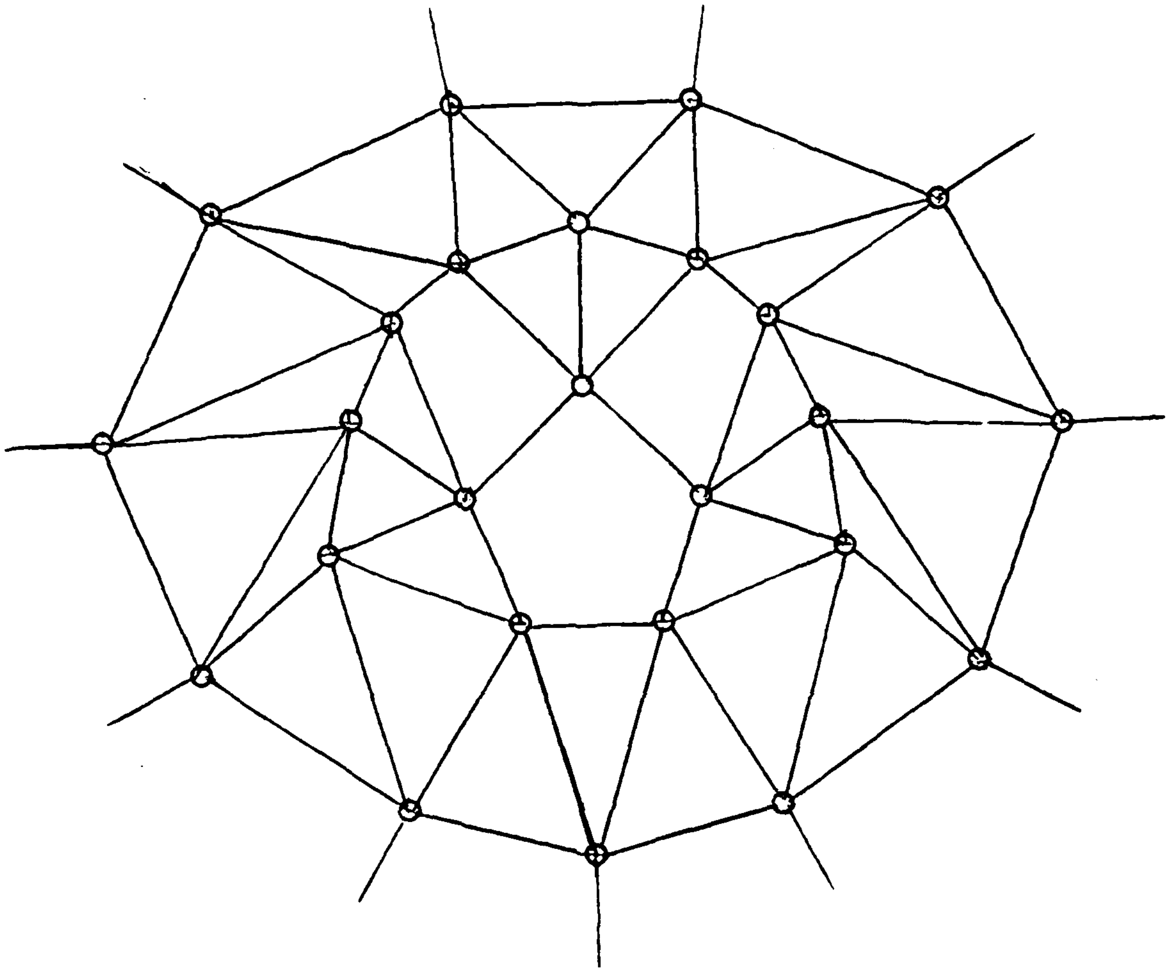


Figure 4. Fragment D

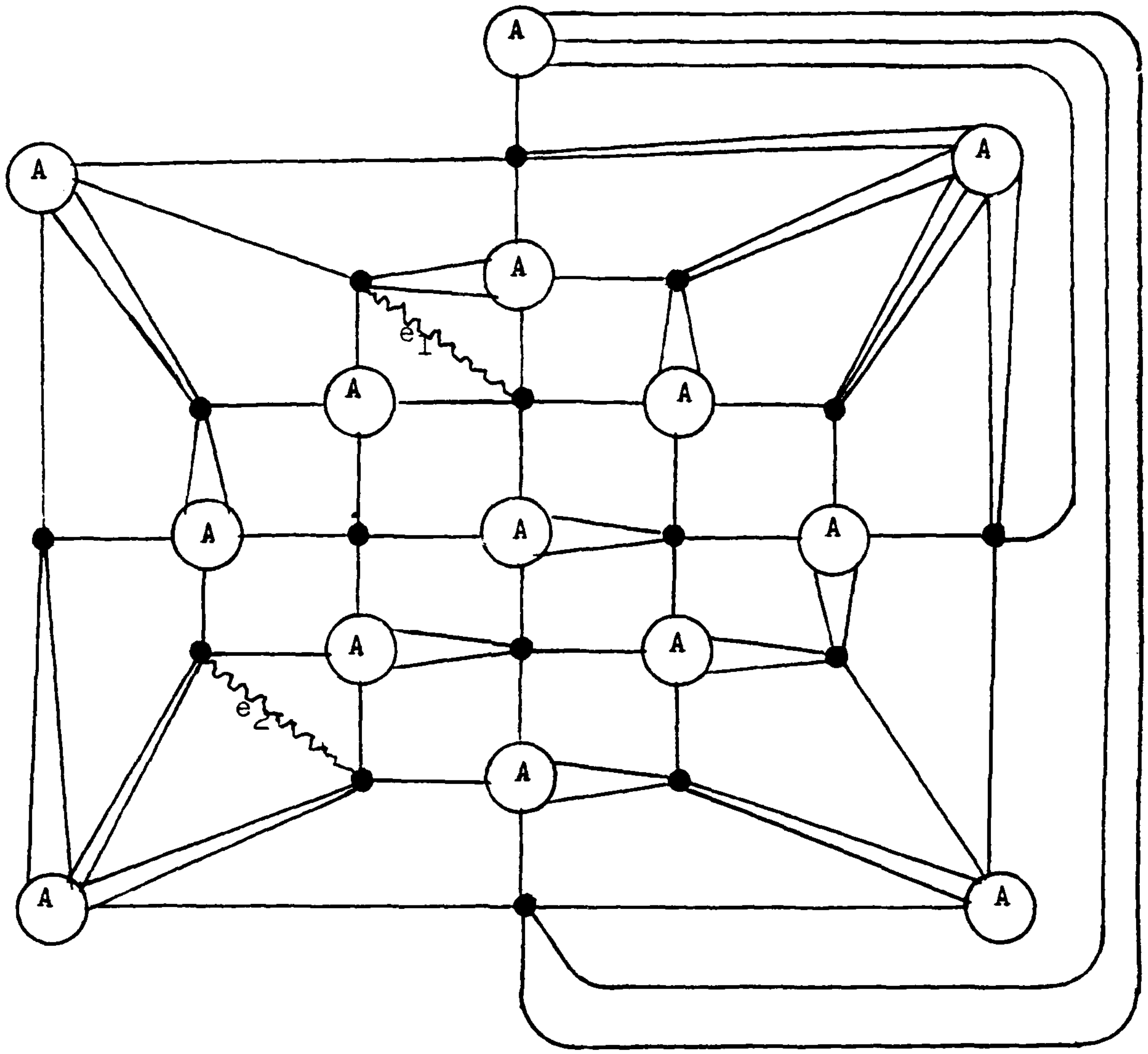


Figure 5. An exceptional graph with 2 lines in G[S]

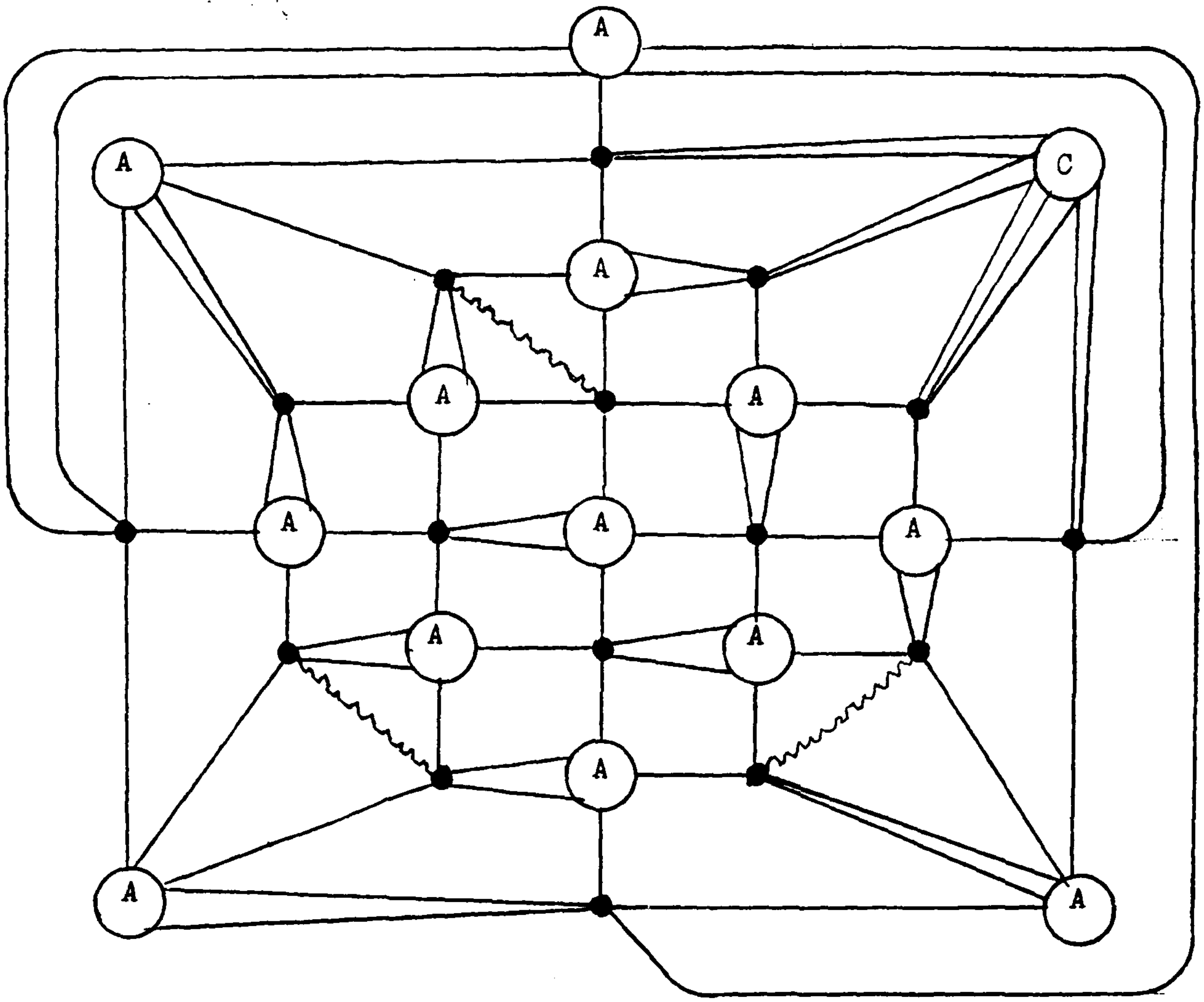


Figure 6. An exceptional graph with 3 lines in G[S]

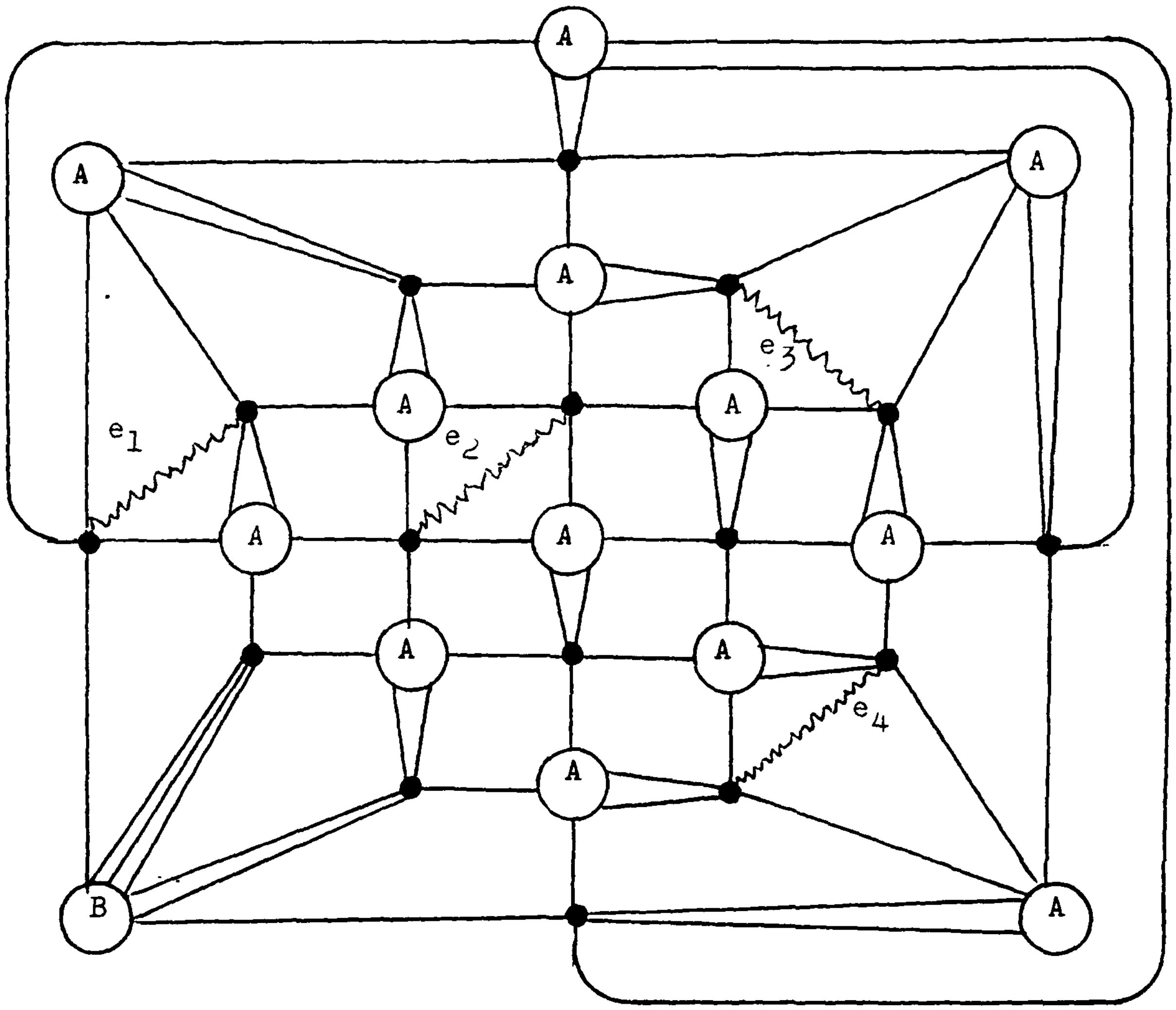


Figure 7. An exceptional graph with 4 lines in G[S]

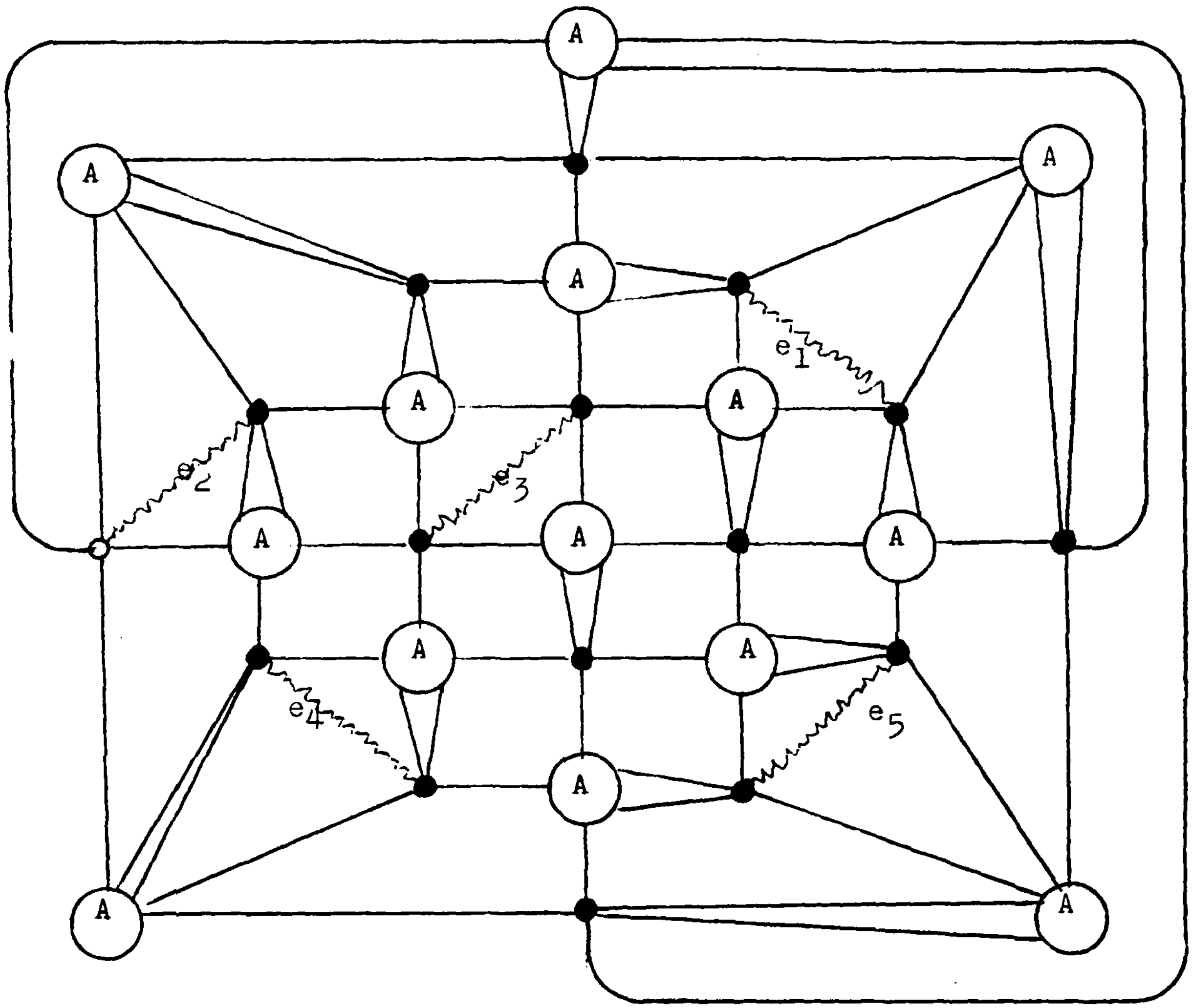
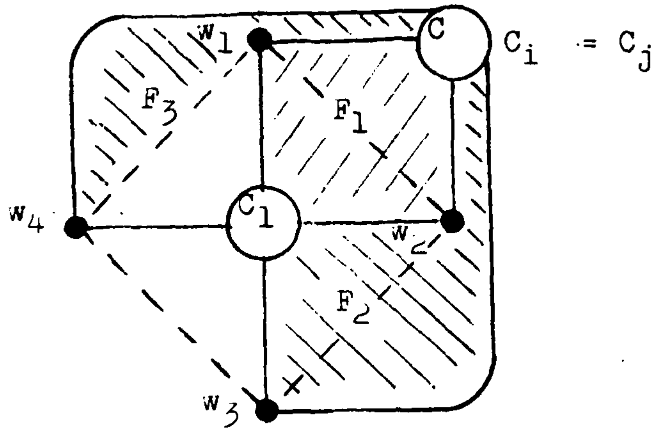
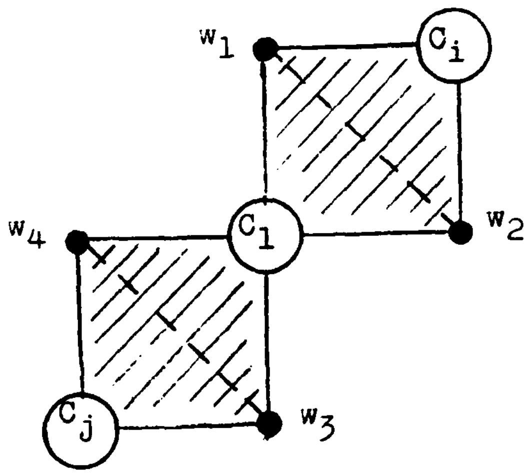


Figure 8. An exceptional graph with 5 lines in G[S]



(a)



(b)

Figure 9.

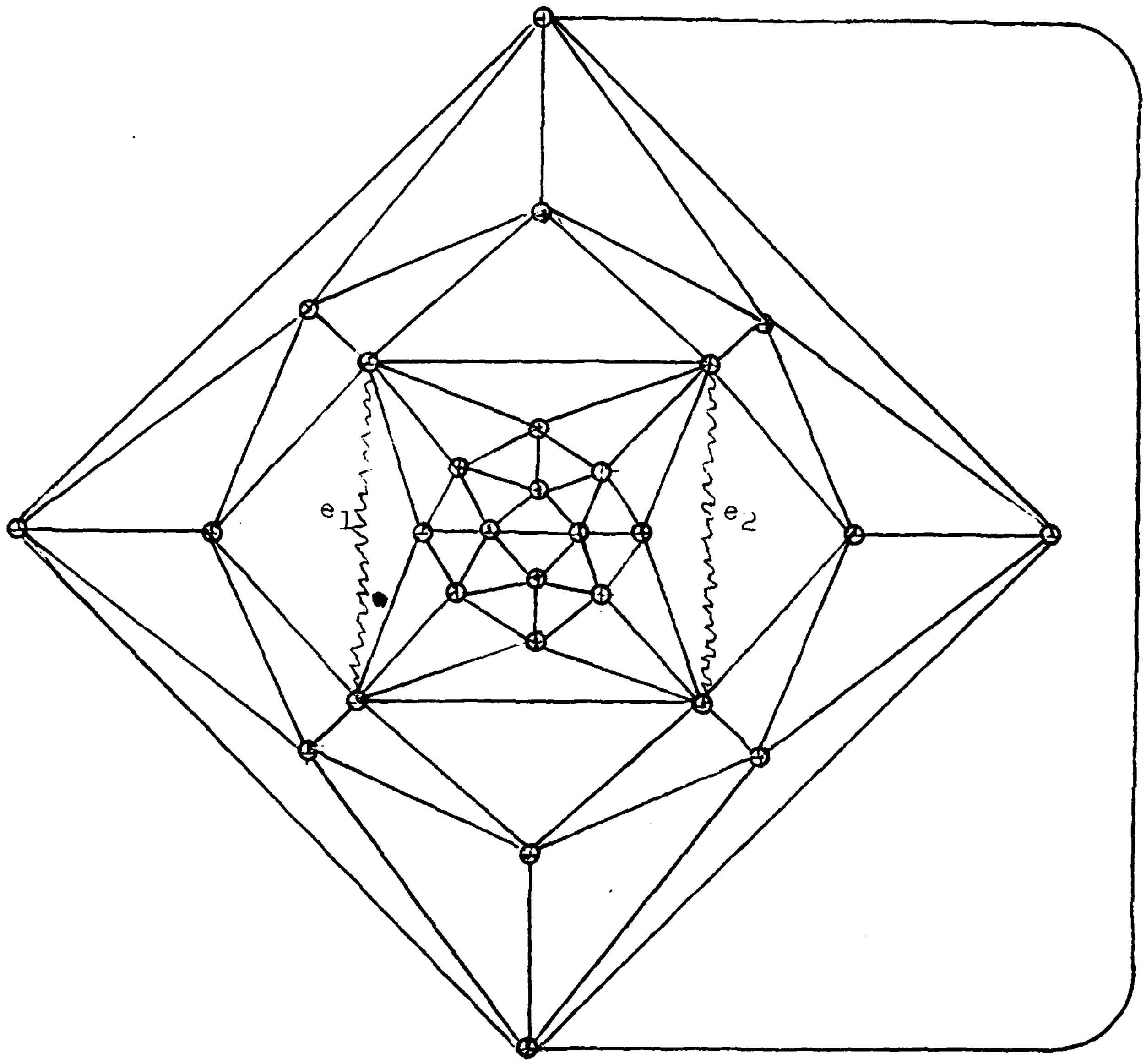
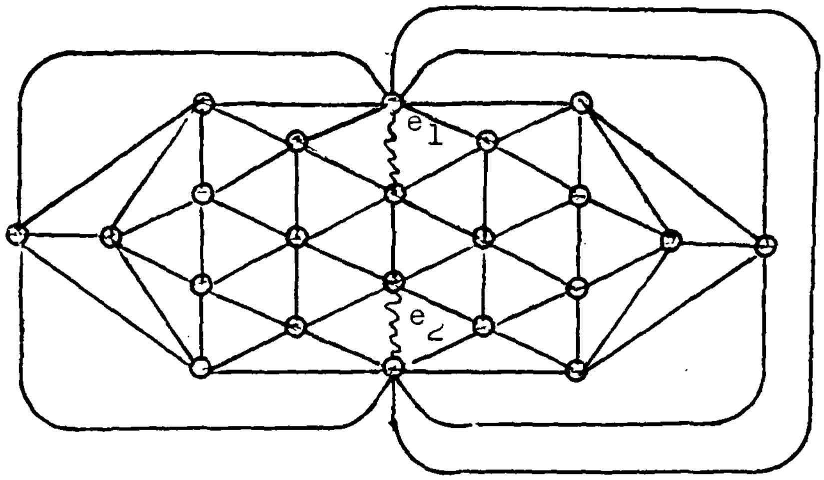
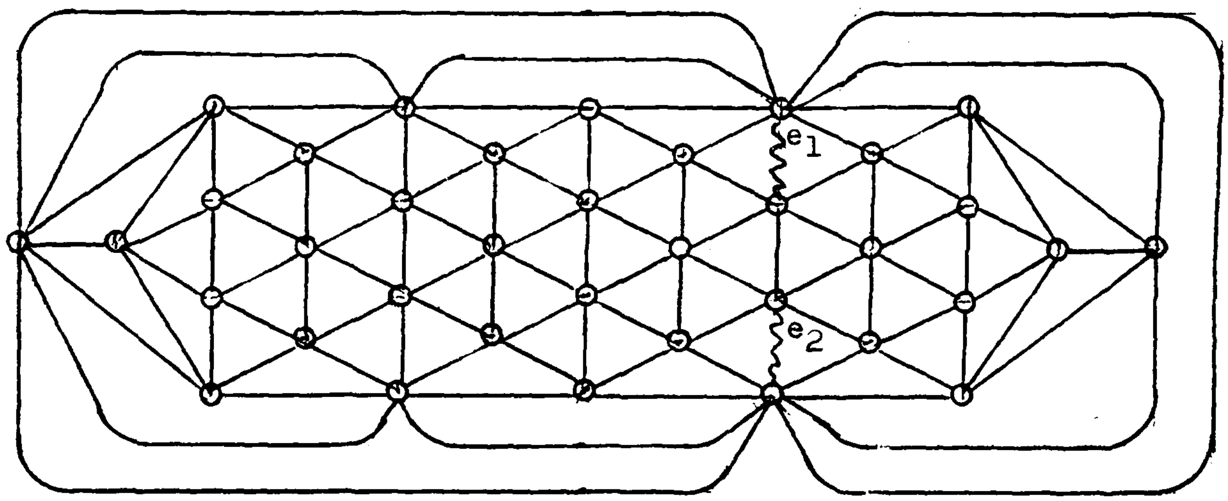


Figure 10.

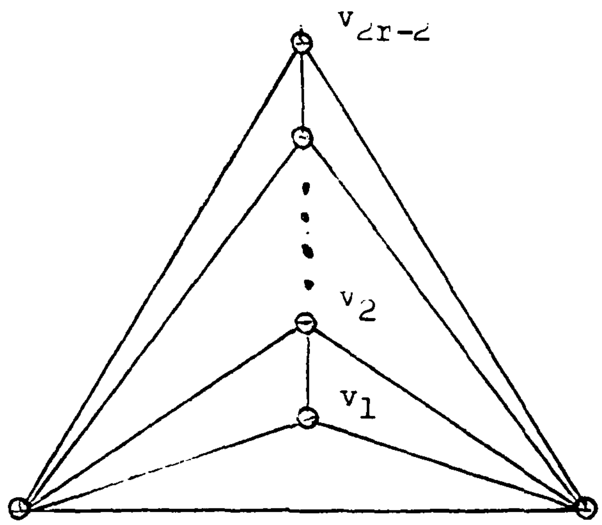


J(22)



J(36)

Figure 11.



T(2r)

Figure 12.

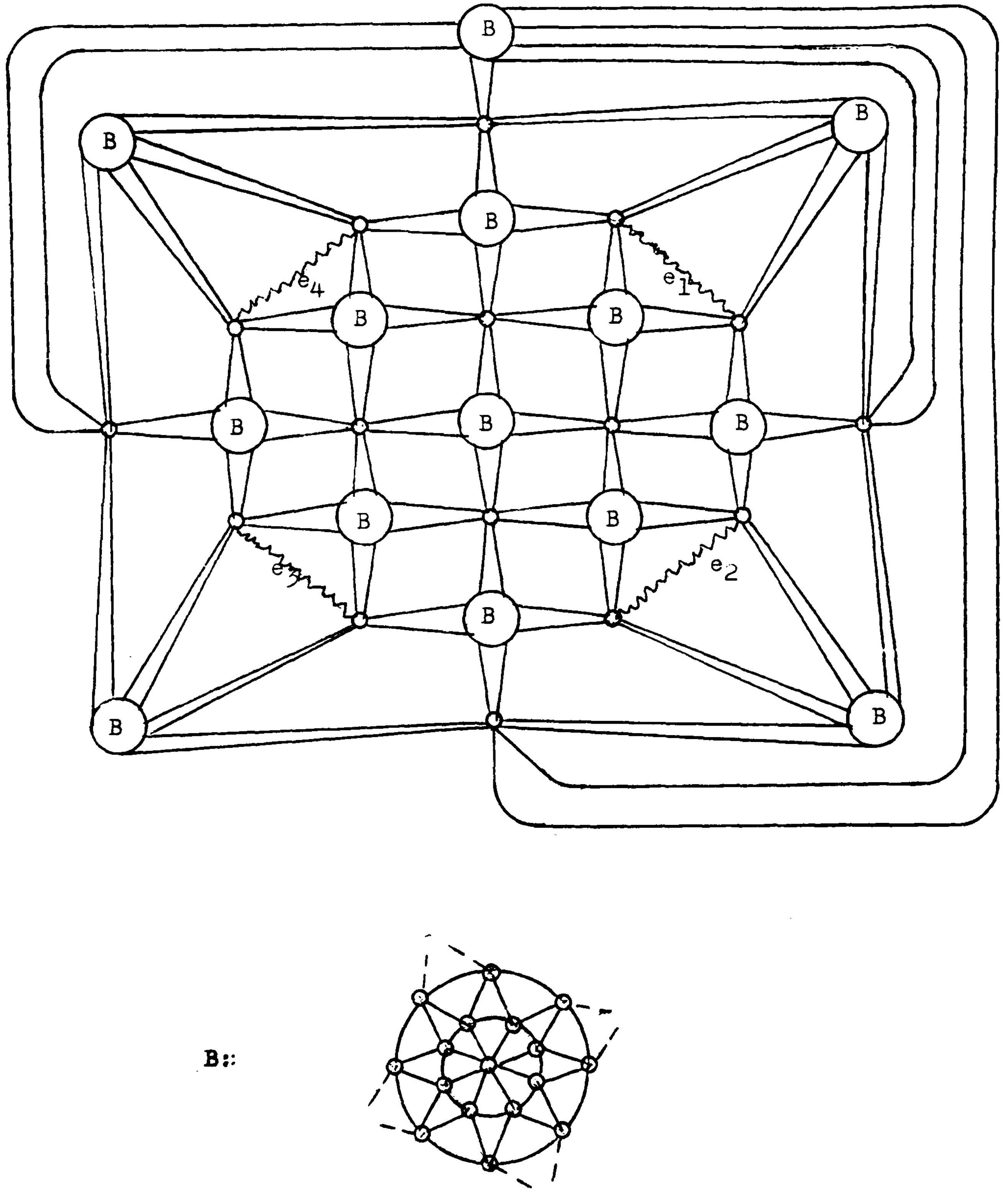


Figure 13.